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A SIMPLE METHOD FOR DETERMINING THE RESPONSE OF LINEAR DYNAMIC SYSTEMS

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ABSTRACT

In this paper, an explicit time integration method to determine the linear response of arbitrary structures subjected to dynamic loading is proposed. The total time of dynamic loading is divided into several time steps. For each two time steps, in modeling the acceleration over time domain, a second order polynomial with three unknown parameters is assumed. Validity and effectiveness of the proposed method is demonstrated through two examples where the results of this method are compared with those numerical methods. In this method, over two steps, six unknown responses (three responses for each time step) consisting of two displacements, two velocities and two accelerations are computed. This property reduces the computational cost of the proposed method as compared to Central difference, Houbolt, Newmark (linear and average), Wilson θ etc. Furthermore, accuracy of the results obtained from the proposed method is better than other methods for single and multi-degree of freedom systems. Hence, as advantages of the proposed method, this method has appropriate convergence, accuracy and low computational time. Therefore, the novelty of this work is that for very small values of Δt , this method is more precise and less time consuming rather than other existed methods. This is a useful instrument for the analysis of dynamic systems with very small values of Δt under earthquake loading.

Keywords: Linear dynamic response analysis; explicit method; accuracy; stability.

1. INTRODUCTION

Solving the differential equation of motion governing single degree of freedom (SDOF) and multi degree of freedom (MDOF) systems can be done through various methods. Since the

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applied loads are not specific mathematical functions, numerical methods seems to be only option for solving such differential equations. Time integration methods are well suited for linear problems in structural dynamics and for dynamic analysis of very large structures in which the equilibrium equations are solved at discrete times. In fact, they are a solution of a set of equilibrium equations over each time step. It can take a significant amount of time to solve structural systems with large degrees of freedom. Thus, today, we are mostly interested in those numerical methods, which not only provide acceptable accuracy and stability, but also solve a problem in fastest possible time. Simulation of a complex dynamic system requires a highly efficient algorithm for time integration, with high accuracy and limited amount of computation. These requirements have attracted many researchers [1-2].

In general, there are two fundemantal classifications for step-by-step integration method. One is explicit [3-7] and the other is implicit [8-12]. In the explicit method, the equation of the current time step is not utilized in determining the current step displacement while it is implicit if that is involved [13]. One of the best advantages of the explicit methods is that in these methods, it is unnecessary to solve a system of equations for each time step, hence less storage is required as compared to the implicit methods [14]. Almost all of the explicit time integration schemes are conditionally stable and a few are unconditionally stable; however, consistency is conditional, which is the major disadvantage of explicit methods. For wave propagation and shock response, the explicit algorithms such as the Central Difference method, the Runge-Kutta method and so on are very efficient since implementation of an explicit method is much simpler than an implicit method for performing pseudo-dynamic tests [3-15].

To determine the response of a dynamical system subjected to the external loading, one can formulated the equations of motion into two domains: 1- in the time domain and 2- frequency domain analysis. It is important to note that the response analysis procedures whether formulated over time domain or the frequency domain, involve evaluation of many independent response contributions that are combined to obtain the total response. In time domain procedures (Duhamel integral), the loading *P(t)* is considered to be a succession of short-duration pulses, and the free vibration response to each pulse becomes a separate contribution to the total response at any subsequent time [16].

Step-by-step procedures are another general approach for dynamic response analysis. There are many different step-by-step methods in which the loading and the response history are calculated in a step from the existing initial conditions (displacement and velocity) at the beginning of the step and the load history during the step. Thus, the response for each step is an independent analysis problem, and there is no need to combine response contributions within the step [16].

In Ref. [17], Heidari and Salajegheh presented an approximate dynamic analysis method to determine the responses of the structure by using Fast Wavelet Transform (FWT). In Ref. [18], Liu et al. presented an efficient time-integration method for obtaining reliable solutions of the transient nonlinear dynamic problems.

In Ref. [19], Rostami et al. presented a scheme where the cubic B-spline method was developed for Multi-Degrees Of-Freedom (MDOF) systems. In this proposed approach, a straightforward formulation in a fluent manner was derived from the approximation of the response of the system with a B-spline basis.

In Refs. [20-22], the method which was proposed by Bathe, Bathe and Baig and Bathe and Noh which is referred to as the Bathe method is presented to calculate the response of linaer and nonlinear dynamical systems with high precision.

A comparison between Bathe and Newmark methods based on dispersion properties of the Bathe method reveals that the desired characteristics obtained by Bathe method in the solution of wave propagation problems are extremely significant [23].

In Ref. [24], Liu employed the piecewise Birkhoff interpolation polynomials and the modal superposition method for the solution of dynamic response of MDOF systems. In this paper, each loading is represented by a piecewise polynomial that reduces the computational effort when is compared to the traditional step-by-step integration solution technique.

In this research, an explicit time integration method is proposed to determine the linear response of arbitrary structures under dynamic loading. In this method, a second order polynomial is used to model acceleration over time domain. The time step is divided into two sub domain and the unknown coefficients in acceleration function are determined by applying proper conditions. Since the time step is divided into two sub domains, hence for each step, there are six increments including two acceleration, two velocity and two displacement increments. Using time integration, the velocity and displacement functions are computed. Consequently, for each step there are two velocity and two displacement increments. Over each step, the acceleration and velocity increments are related to the two displacement increments as well as the acceleration, velocity and displacement values at beginning of the time step, which are known. Hence, by constructing a system of equations and solving them to determine the displacement increments, all unknown values are computed. This method is presented for SDOF and MDOF systems. It is important to note that all of the methods are compared based on developed programs in MATLAB software and run by the same computer. Also, for all different methods, the calculation time to compute the responses of the structure have been measured without considering the required time for reading the input data and are compared with each other. Validity and effectiveness of the proposed method are demonstrated with two examples for which the results of this method are compared to those from other existing numerical methods. In this method, for each step, six unknown responses consisting of two displacements, two velocities and two accelerations are computed.

2. PROGRAM FORMULATION AND SOLUTION

The equations of motion for MDOF systems are most easily formulated by directly expressing the equilibrium of all forces acting on the masses using D'Alembert's principle. In the case of MDOF systems and linear dynamic analysis, it can be written in the matrix form as follows:

$$[M]\{\ddot{X}\}+[C]\{\dot{X}\}+[K]\{X\}=\{P\} \tag{1}$$

where $[M],[C],[K],\{\ddot{X}\},\{\dot{X}\},\{X\}$ and $\{P\}$ are respectively the mass matrix, damping matrix, lateral stiffness matrix, acceleration vector, velocity vector, displacement vector and external load vector for a MDOF system. For the sake of simplicity, the structure is considered as a SDOF system. The incremental equilibrium equation at time *t* can be written as follows:

$$m\,\Delta\ddot{x} + c\,\Delta\dot{x} + k\,\Delta x = \Delta P \tag{2}$$

The step-by-step integration procedures can now be modified into an incremental form. Lets assume that distribution of acceleration is a second order polynomial function over two time steps as shown in Fig. 1.

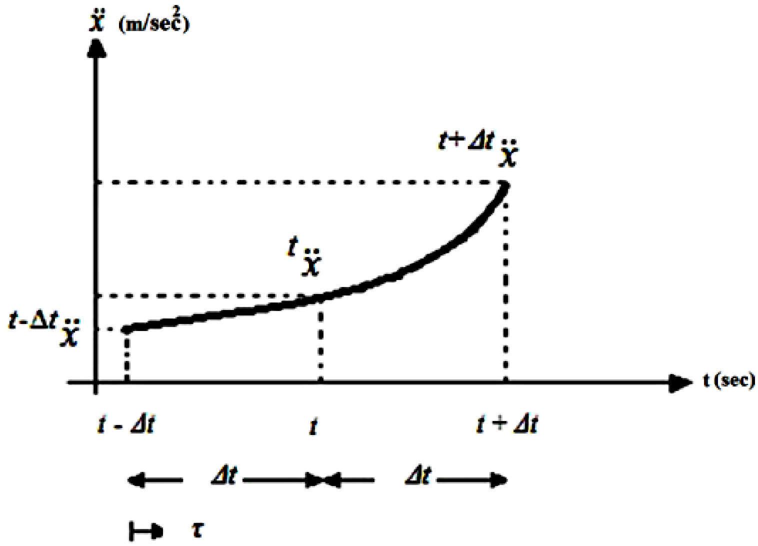

Figure 1. Distribution of acceleration over two time steps

Thus, the equation for acceleration over two time steps can be written as follows:

$$\ddot{x}(\tau) = a\tau^2 + b\tau + c \qquad 0 \le \tau \le 2\,\Delta t \tag{3}$$

By substituting the conditions at times $t - \Delta t$, t and $t + \Delta t$ into Eq. (3), the constants *a*, *b* and *c* can be determined as follows:

$$a = \frac{^{t-\Delta t}\ddot{x} - 2\,{}^{t}\ddot{x} + {}^{t+\Delta t}\ddot{x}}{2\left(\Delta t\right)^2}, b = \frac{4\,{}^{t}\ddot{x} - 3\,{}^{t-\Delta t}\ddot{x} - {}^{t+\Delta t}\ddot{x}}{2\left(\Delta t\right)} \tag{4}$$
$$c = {}^{t-\Delta t}\ddot{x}$$

By integrating over time domain from Eq. (3), and considering relations (${}^{t}\dot{x} = {}^{t-\Delta t}\dot{x} + \Delta\dot{x}_{(1)}, 0 \le \tau \le \Delta t$ and ${}^{t+\Delta t}\dot{x} = {}^{t}\dot{x} + \Delta\dot{x}_{(2)}, \Delta t \le \tau \le 2\,\Delta t$), one can obtain:

$$\begin{aligned} \Delta\dot{x}_{(1)} &= \frac{a}{3}(\Delta t)^3 + \frac{b}{2}(\Delta t)^2 + c(\Delta t) \\ \Delta\dot{x}_{(2)} &= \frac{7a}{3}(\Delta t)^3 + \frac{3b}{2}(\Delta t)^2 + c(\Delta t) \end{aligned} \tag{5}$$

By considering relations $({}^{t}x = {}^{t-\Delta t}x + \Delta x_{(1)}, 0 \le \tau \le \Delta t$ and ${}^{t+\Delta t}x = {}^{t}x + \Delta x_{(2)}, \Delta t \le \tau \le 2\Delta t)$, the same procedure is used for displacement and the following equations are determined:

$$\begin{aligned} \Delta x_{(1)} &= \frac{a}{12}(\Delta t)^4 + \frac{b}{6}(\Delta t)^3 + \frac{c}{2}(\Delta t)^2 + {}^{t-\Delta t}\dot{x}(\Delta t) \\ \Delta x_{(2)} &= \frac{15a}{12}(\Delta t)^4 + \frac{7b}{6}(\Delta t)^3 + \frac{3c}{2}(\Delta t)^2 + {}^{t-\Delta t}\dot{x}(\Delta t) \end{aligned} \tag{6}$$

Considering Eq. (6) and the relations for acceleration $({}^{t}\ddot{x} = {}^{t-\Delta t}\ddot{x} + \Delta\ddot{x}_{(1)}, 0 \le \tau \le \Delta t$ and ${}^{t+\Delta t}\ddot{x} = {}^{t}\ddot{x} + \Delta\ddot{x}_{(2)}, \Delta t \le \tau \le 2\Delta t)$, $\Delta\ddot{x}_{(1)}, \Delta\ddot{x}_{(2)}, \Delta\dot{x}_{(1)}$ and $\Delta\dot{x}_{(2)}$ can be obtained as follows:

$$\begin{aligned} \Delta\ddot{x}_{(1)} &= \frac{3}{4\Delta t^2}\left(\Delta x_{(1)} + \Delta x_{(2)}\right) - \frac{3}{2}\left({}^{t-\Delta t}\ddot{x} + \frac{{}^{t-\Delta t}\dot{x}}{\Delta t}\right) \\ \Delta\ddot{x}_{(2)} &= \frac{15}{4\Delta t^2}\Delta x_{(2)} - \frac{81}{4\Delta t^2}\Delta x_{(1)} + \frac{9}{2}\,{}^{t-\Delta t}\ddot{x} + \frac{33}{2\Delta t}\,{}^{t-\Delta t}\dot{x} \\ \Delta\dot{x}_{(1)} &= \frac{\Delta t}{6}\left(\Delta\ddot{x}_{(2)} - \Delta\ddot{x}_{(1)}\right) + \frac{\Delta t}{4}\left(3\Delta\ddot{x}_{(1)} - \Delta\ddot{x}_{(2)}\right) \\ &\quad + \Delta t\,{}^{t-\Delta t}\ddot{x} \\ \Delta\dot{x}_{(2)} &= \frac{7\Delta t}{6}\left(\Delta\ddot{x}_{(2)} - \Delta\ddot{x}_{(1)}\right) + \frac{3\Delta t}{4} \times \left(3\Delta\ddot{x}_{(1)} - \Delta\ddot{x}_{(2)}\right) \\ &\quad + \Delta t\,{}^{t-\Delta t}\ddot{x} \end{aligned} \tag{7}$$

Using Eq. (7), the parameters $\left(\Delta\ddot{x}_{(2)} - \Delta\ddot{x}_{(1)}\right)$ and $\left(3\,\Delta\ddot{x}_{(1)} - \Delta\ddot{x}_{(2)}\right)$ can be computed and then by substituting them into Eq. (1), for an arbitrary system of MDOF, the following relation can be obtained in matrix form as follows:

$$
\begin{bmatrix} \hat{K}_{11} & \hat{K}_{12} \\ \hat{K}_{21} & \hat{K}_{22} \end{bmatrix}_{2N\times 2N} \begin{Bmatrix} \left\{\Delta x_{(1)}\right\}_{N\times 1} \\ \left\{\Delta x_{(2)}\right\}_{N\times 1} \end{Bmatrix}_{2N\times 1} = \begin{Bmatrix} \left\{\Delta \hat{P}_{(1)}\right\} \\ \left\{\Delta \hat{P}_{(2)}\right\} \end{Bmatrix}_{2N\times 1}
$$

$$
\begin{aligned}
\hat{K}_{11} &= \frac{3}{4\Delta t^2}\left[M\right]_{N\times N} + \frac{17}{8\Delta t}\left[C\right]_{N\times N} + \left[K\right]_{N\times N} \\
\hat{K}_{12} &= \frac{3}{4\Delta t^2}\left[M\right]_{N\times N} + \frac{1}{8\Delta t}\left[C\right]_{N\times N} \\
\hat{K}_{21} &= -\frac{81}{4\Delta t^2}\left[M\right]_{N\times N} - \frac{61}{8\Delta t}\left[C\right]_{N\times N} \\
\hat{K}_{22} &= \frac{15}{4\Delta t^2}\left[M\right]_{N\times N} + \frac{19}{8\Delta t}\left[C\right]_{N\times N} + \left[K\right]_{N\times N}
\end{aligned}
\tag{8}
$$

where N is the number of degrees of freedom in the system being studied and:

$$
\begin{aligned}
\left\{\Delta \hat{P}_{(1)}\right\}_{N\times 1} &= \left\{\Delta P_{(1)}\right\} + \frac{3}{2}\left[M\right]\left\{{}^{t\text{-}\Delta t}\ddot{x}\right\} + \frac{3}{2\Delta t}\left[M\right]\left\{{}^{t\text{-}\Delta t}\dot{x}\right\} \\
&\quad + \frac{9}{4}\left[C\right]\left\{{}^{t\text{-}\Delta t}\dot{x}\right\} + \frac{\Delta t}{4}\left[C\right]\left\{{}^{t\text{-}\Delta t}\ddot{x}\right\} \\
\left\{\Delta \hat{P}_{(2)}\right\}_{N\times 1} &= \left\{\Delta P_{(2)}\right\} - \frac{9}{2}\left[M\right]\left\{{}^{t\text{-}\Delta t}\ddot{x}\right\} - \frac{33}{2\Delta t}\left[M\right]\left\{{}^{t\text{-}\Delta t}\dot{x}\right\} \\
&\quad - \frac{21}{4}\left[C\right]\left\{{}^{t\text{-}\Delta t}\dot{x}\right\} - \frac{5\Delta t}{4}\left[C\right]\left\{{}^{t\text{-}\Delta t}\ddot{x}\right\} \\
\left\{\Delta P_{(1)}\right\}_{N\times 1} &= \left\{P_{(t)}\right\} - \left\{P_{(t\text{-}\Delta t)}\right\} \\
\left\{\Delta P_{(2)}\right\}_{N\times 1} &= \left\{P_{(t+\Delta t)}\right\} - \left\{P_{(t)}\right\}
\end{aligned}
\tag{9}
$$

To compute the response of a MDOF system, the values of $\left\{\Delta \hat{P}_{(1)}\right\}$ and $\left\{\Delta \hat{P}_{(2)}\right\}$ can be determined using Eq. (9) and therefore by replacing them into Eq. (8), the incremental displacements can be computed. Finally by using Eq. (7), the incremental values of velocity and acceleration will be computed and replaced into Eq. (10) to compute the values of displacement, velocity and acceleration at times t and $t+\Delta t$.

$$
\begin{aligned}
{}^{t}x &= {}^{t\text{-}\Delta t}x + \Delta x_{(1)} \qquad & {}^{t+\Delta t}x &= {}^{t}x + \Delta x_{(2)} \\
{}^{t}\dot{x} &= {}^{t\text{-}\Delta t}\dot{x} + \Delta \dot{x}_{(1)} \qquad & {}^{t+\Delta t}\dot{x} &= {}^{t}\dot{x} + \Delta \dot{x}_{(2)} \\
{}^{t}\ddot{x} &= {}^{t\text{-}\Delta t}\ddot{x} + \Delta \ddot{x}_{(1)} \qquad & {}^{t+\Delta t}\ddot{x} &= {}^{t}\ddot{x} + \Delta \ddot{x}_{(2)}
\end{aligned}
\tag{10}
$$

3. EVALUATION STABILITY OF THE PROPOSED METHOD

In direct integration, Eq. (1) is integrated using a numerical step-by-step procedure, the term “direct” implies that prior to numerical integration, there is no transformation of the equations into a different form. In essence, direct numerical integration is based on two ideas. First, instead of trying to satisfy Eq. (1) for all time, it tries to satisfy Eq. (1) at discrete time points by Δt increments. The second idea for a direct integration method is based on a variation for displacements, velocities, and accelerations within each time interval. The form of variations in displacement, velocity, and acceleration determines the accuracy, stability, and cost of the solution procedure [13].

It is assumed that the displacement $\left\{{}^{0}x\right\}$, velocity $\left\{{}^{0}\dot{x}\right\}$, and acceleration $\left\{{}^{0}\ddot{x}\right\}$ vectors at time $t=0$ are known and the solution of Eq. (1) is required to be obtained from time 0 to time T. Since the algorithm calculates the solution at the next required time from the solution over previous time interval, it is assumed that the solutions at times 0, Δt , $2\Delta t$, $3\Delta t$, …, $t - \Delta t$ are known and the solution at times t and $t + \Delta t$ is required next. Then for the proposed integration method, it is desirable to establish the following recursive relationship:

$$ {}^{t+\Delta t}\hat{X} = A\ {}^{t-\Delta t}\hat{X} + L\left({}^{t+v}r\right) \tag{11}$$

where ${}^{t-\Delta t}\hat{X}$ and ${}^{t+\Delta t}\hat{X}$ are vectors storing the solution quantities (displacements, velocities and accelerations) and ${}^{t+v}r$ is the load at time $t+v$. Matrix A and vector L are the integration approximation and load operator, respectively. Also matrix A is known as the amplified matrix [13].

Stability of an integration method can be determined by examining the results of a numerical solution for different initial conditions [13].

The equilibrium equation of an arbitrary system with SDOF at time $(t+\Delta t)$ can be written as follows:

$$ {}^{t+\Delta t}\ddot{x} + 2\,\zeta\,w_n\ {}^{t+\Delta t}\dot{x} + w_n^2\ {}^{t+\Delta t}x = \frac{{}^{t+\Delta t}P}{m} \tag{12}$$

It is be noted that for evaluation the stability of time integration method, it is assumed that there is no load on the SDOF system ($P=0$) and therefore as a result $r=0$ [13].

To investigate stability of the proposed method, at first, it is required to find the amplification matrix, thus constructing a relation in which the values of responses at the end of each time step are written in terms of values at the beginning of time step. For stability analysis, one examines eigenvalues of the approximation operator or amplification matrix A; which in general is a non-symmetric matrix.

Over each time step, ($t - \Delta t$ to $t + \Delta t$), by substituting Eqs. (7), (8) and (10) into Eq. (12), and assuming that $\zeta = 0$, (damping ratio $\zeta = \dfrac{C}{C_{cr}} = \dfrac{C}{2m\omega_n}$) one obtains:

$$\begin{Bmatrix} {}^{t+\Delta t}x \\ {}^{t+\Delta t}\dot{x} \end{Bmatrix} = \begin{bmatrix} L_1 & L_2 \\ L_3 & L_4 \end{bmatrix} \begin{Bmatrix} {}^{t-\Delta t}\dot{x} \\ {}^{t-\Delta t}\ddot{x} \end{Bmatrix} + L^{t+v}r \tag{13}$$

where:

$$\begin{aligned} L_1 &= -\frac{15m}{\Delta tk} + \frac{2160\ m^3}{\Delta t\ I^*k} + \frac{828\ m^2\Delta t}{I^*} \\ L_2 &= -\frac{4m}{k} + \frac{432\ m^3}{I^*k} + \frac{396\ m^2\Delta t^2}{I^*} \\ L_3 &= -\frac{432\ m^2}{I^*} - \frac{792\ k\ m\ \Delta t^2}{2\ I^*} + 4 \\ L_4 &= \frac{144\ m^2\ \Delta t}{I^*} - \frac{312\ k\ m\ \Delta t^3}{2\ I^*} + \Delta t \\ I^* &= 8\ k^2\Delta t^4 + 36\ k\ m\ \Delta t^2 + 144\ m^2 \end{aligned} \tag{14}$$

The integration of Eq. (14) is needed when the load is not specified; i.e. $\boldsymbol{r}$ =0 [13]. Stability analysis can be performed by solving for the eigenvalues of amplification matrix. The eigenvalues and eigenvectors of *A* are calculated using $\left(\left[A\right] - \lambda\left[I\right]\right)\left\{\Phi\right\} = \left\{0\right\}$. It is now possible to write *A* in terms of its eigenvalues and eigenvectors $\left[A\right] = \left[\lambda\right]\left[\Phi\right]\left[\Phi\right]^{-1}$. Here, columns of $\left[\Phi\right]$ are eigenvectors of *A*, and $\left[\lambda\right]$ is a diagonal matrix holding eigenvalues of *A*.

$$det\left(\begin{bmatrix} L_1 & L_2 \\ L_3 & L_4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0 \tag{15}$$

From Eq. (15), one can obtain:

$$a'\lambda^2 + b'\lambda + c' = 0 \tag{16}$$

where:

$$a' = 1.0 \quad c' = \frac{1}{\omega_n^2}$$

$$b' = \Delta t \left(\frac{-2\,\omega_n^4 \Delta t^4 + 60\,\omega_n^2 \Delta t^2 - 144}{2\,\omega_n^4 \Delta t^4 + 9\,\omega_n^2 \Delta t^2 + 36} \right)$$

Now, in order to have a stable solution, norm of every element in $[\lambda]$ should not be greater than unity;

$$\rho(A) = max\left(\|\lambda_1\|, \|\lambda_2\|\right) \le 1 \tag{17}$$

where:

$$\lambda_1 = \frac{-b' + \sqrt{\Delta}}{2a'} \qquad \lambda_2 = \frac{-b' - \sqrt{\Delta}}{2a'} \tag{18-a}$$

and

$$\Delta = \frac{\left(4\,\omega_n^8 \Delta t^{10} - 256\,\omega_n^6 \Delta t^8 + 4032\,\omega_n^4 \Delta t^6\right)}{\hat{I}}$$
$$\frac{\left(-18180\,\omega_n^2 \Delta t^4 + 18144 \Delta t^2 - \frac{5184}{\omega_n^2}\right)}{\hat{I}} \tag{18-b}$$
$$\hat{I} = 4\,\omega_n^8 \Delta t^8 + 36\,\omega_n^6 \Delta t^6 + 225\,\omega_n^4 \Delta t^4$$
$$+ 648\,\omega_n^2 \Delta t^2 + 1296$$

In Eq. (17), $\rho(A)$ is the spectral radius, which is a function of time step length (Δt) and properties of the system such as *k* and *m*. Even though the spectral radius slightly changes with variations in damping ratio, the damping ratio is assumed to be zero in constructing the amplification matrix.

Considering Eqs. (16), (18-a) and (18-b), coefficients a' and c' are always positive, furthermore, value of the spectral radius is a function of the value and sign of coefficient b' and $\Delta = b'^2 - (4\,a'\,c')$ parameter.

Since T_n is the natural period in SDOF systems and the lowest natural period in MDOF systems, hence, ω_n is the undamped natural circular frequency in SDOF systems and the largest undamped natural circular frequency in MDOF systems. These parameters are related through $\omega_n = 2\pi / T_n$.

In addition, considering Eq. (18-b), the denominator is always a positive value, hence, the roots and sign of the Δ parameter is a function of the numerator. As can be seen from Eq. (18-b), the roots of Δ parameter is a function of structural property (ω_n) and incremental time (Δt). Based on this formulation, it is difficult to find all roots of Δ parameter and signs of them over the positive domain of real numbers. By having the largest natural frequency for a given structural system, it is easy to find the positive real roots of the numerator of Eq. (18-b) (which is a polynomial of order ten) hence, one can determine the sign of Δ parameter over the domain of positive real numbers. For Δ parameter, two of the positive real roots are always $\left(0.7913 / \omega_n\right)$ and $\left(3.7913 / \omega_n\right)$.

Thus in general, by having the maximum natural frequency of a structure, and by using Eq. (18-b), one can obtain other roots and sign for the Δ parameter over different time intervals.

By considering Eqs. (16-18), the spectral radius can be calculated and the maximum Δt is obtained so that the spectral radius is less than one. Now, one can plot the values of spectral radius for the various structural properties (ω_n) and various incremental times. Also, those domains whit values less than one can be computed. The spectral radius can be calculated from Figs. 2a and 2b.

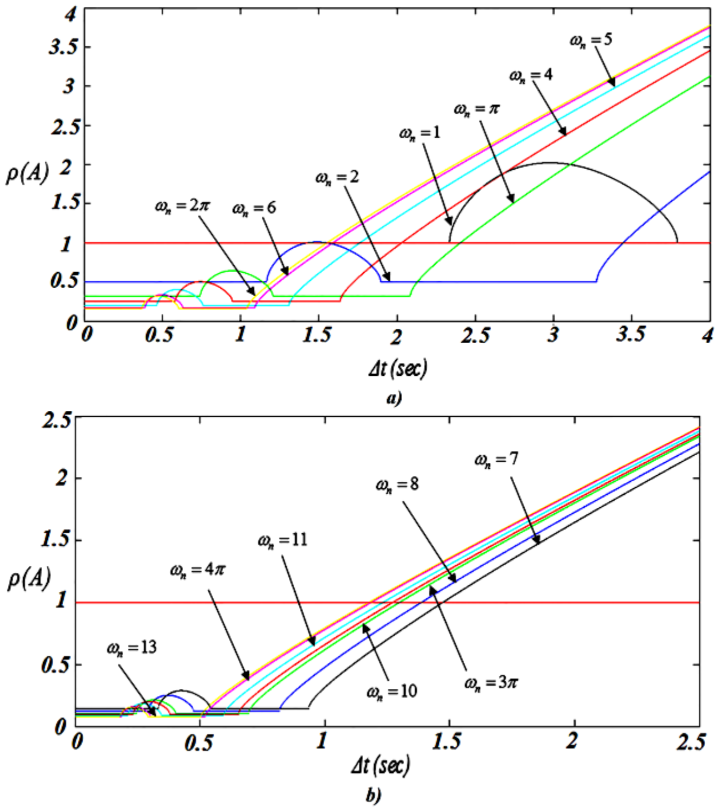

Figure 2. The values of spectral radius of proposed method for various natural frequency between a) $\omega_n = 1$ to 2π and b) $\omega_n = 7$ to 13

4. EVALUATION OF ERRORS FOR THE PROPOSED METHOD

There are undeniable errors in the any numerical solution of the equation of motion [26]. For a numerical step by step procedure, quantify the difference in computed displacements with the exact displacements in a free vibration problem, is a common method that is used to gain insight into the magnitude of error. An undamped SDOF system under initial displacement x_0 and initial velocity $\dot{x}_0$ has a cyclic response with a constant maximum amplitude $|x(t)|_{max}$ and constant period T_0 which is equal to $2\pi/\omega$. The maximum amplitude is equal to:

$$|x(t)|_{max} = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega}\right)^2} \tag{19}$$

There is two definitions of error for free vibration problems: (a) amplitude decay and (b) period elongation [13]. Since the SDOF systems are assumed to be undamped, therefore any computed displacement by using a numerical method is named as computed error. This error is reported as 'algorithmic damping' [2]. In this paper, amplitude decay is called the algorithmic damping. The second type of error is referred to as period elongation and measures the extension in the time period which takes to complete each cycle of harmonic response [26].

In this way to investigate accuracy of the proposed method, an undamped SDOF system with the considered initial condition is assumed as follows:

$$\begin{cases} \ddot{x}(t) + \omega^2 x(t) = 0, \\ x_0 = 1; \ \dot{x}_0 = 0; \ \ddot{x}_0 = -\omega^2 \end{cases} \tag{20}$$

The exact solution of Eq. (20) is $x = cos\,(\omega \times t)$. The response is calculated through a numerical method and compared to the response that is obtained from the exact solution. Therefore, the two types of errors are calculated based on $\Delta t/T_0$ variations. For comparison, the results are shown in Fig. 3. This type of Figures can be found in many references such as (Bathe [13], Chopra [26], Kelly [27], Hughes [28] and Rostami et al. [29]). In general, the data in Fig. 3 shows that for integration methods, the magnitude of one or both error measurements usually rises with an increase in time-step Δt [28]. Meanwhile, for the specified time step Δt, the magnitude of one or both error measurements is greater for short-period SDOF systems rather than the long-period SDOF systems [29]. As graphs show, the proposed method shows a low rate for both types of errors.

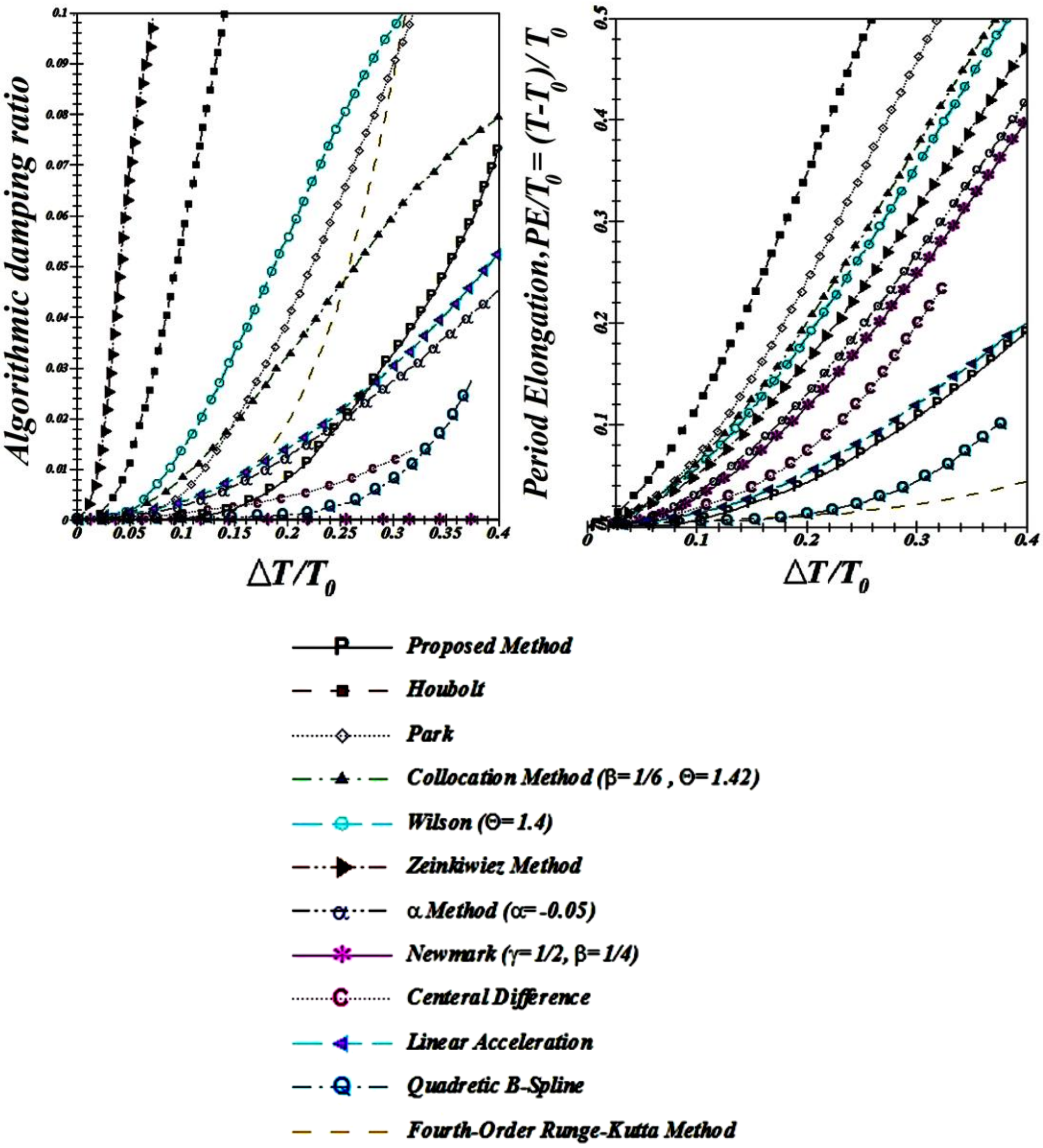

Figure 3. Accuracy investigation of amplitude decay and period elongation

5. EVALUATION ACCURACY OF THE PROPOSED METHOD

In this section, validity of the proposed method is confirmed with examination of several results. Two examples for lumped mass structures are considered, including a SDOF and a two dimensional shear building. To illustrate the accuracy of the proposed method, at first, an SDOF system with following properties is considered: m=0.2533 ($N.sec^2/m$), k=10 (N/m), T_n=1 (sec) (w_n=6.2832 (rad/sec)), and $\zeta = 0.05$. Response of the system in terms of displacement and acceleration are determined when the system is subjected to the

$P(t) = 10\, sin(5\pi t/3)\, (N)$ (a half-cycle sine pulse force) by (a) using the proposed method with $\Delta t = 0.1(sec)$, (b) evaluating closed form solution and (c) comparison to other numerical time integration methods such as Houbolt, Central difference and Newmark (linear and average) method with $\Delta t = 0.1(sec)$. The value of Δt is considered to be the same for all methods to investigate and show the accuracy and precision of the proposed method. Results are shown in Tables 1 and 2. As shown in Tables 1 and 2, the results of the proposed method is fine rather than the other methods with respect to the theoretical values.

Table 1: Differences between the methods in evaluation of displacement (*m*)

Time (sec)	**Newmark (linear acceleration)**	**Newmark (average acceleration)**	**Central difference**	**Houbolt**	**Proposed method**	**Theoretical**	**% error in proposed method with theoretical method**
0	0	0	0	0	0	0	0
0.1	0.02998	0.04367	0.19138	0.02998	0.03482	0.03280	-6.16
0.2	0.21933	0.23262	0.62933	0.21933	0.23655	0.23317	-1.45
0.3	0.61661	0.61207	1.18248	0.56177	0.64912	0.64874	-0.0587
0.4	1.11302	1.08254	1.58081	0.9519	1.15827	1.16049	0.19
0.5	1.47821	1.43095	1.54117	1.21922	1.51956	1.5241	0.30
0.6	1.46249	1.42308	0.91405	1.19608	1.47617	1.48135	0.35
0.7	0.95143	0.96218	-0.02474	0.87089	0.92562	0.92451	-0.12
0.8	0.12730	0.19078	-0.89687	0.34514	0.06570	0.05931	-10.76
0.9	-0.69543	-0.60438	-1.37258	-0.22061	-0.76685	-0.77515	1.07
1	-1.22083	-1.14420	-1.29394	-0.66479	-1.26461	-1.27183	0.57

Table 2: Differences between the methods in evaluation of acceleration (m/sec^2)

Time (sec)	**Newmark (linear acceleration)**	**Newmark (average acceleration)**	**Central difference**	**Houbolt**	**Proposed method**	**Theoretical**	**% error in proposed method with theoretical method**
0	0	0	0	0	0	0	0
0.1	17.99051	17.46678	19.13820	17.99051	17.74152	17.84350	0.57153
0.2	23.65708	23.18047	24.65705	23.65708	22.88522	23.01255	0.55329
0.3	12.13768	12.37236	11.51982	14.68134	10.76068	10.74624	-0.13436
0.4	-12.73042	-11.51736	-15.4826	-5.77066	-14.5787	-14.6669	0.60143
0.5	-39.94333	-38.16181	-43.796	-29.3309	-41.4687	-41.6755	0.49604
0.6	-56.04642	-54.67381	-58.7491	-45.8111	-56.3588	-56.5438	0.327120

0.7	-33.07080	-33.70149	-31.1661	-31.3653	-31.8001	-31.7017	-0.31041
0.8	0.48835	-2.12205	6.66589	-9.905	3.10449	3.36410	7.717145
0.9	31.95001	28.44295	39.64176	12.05375	34.70763	35.04895	0.97383
1	50.1141	47.37246	55.43537	28.31582	51.57090	51.82845	0.49693

As a second example, a lumped mass model of a nine storey shear building shown in Fig. 4 is considered. This building is subjected to sinusoidal base acceleration with frequency of 5π (*rad/sec*) and *PGA=0.3g*. Columns are I-shaped with sections of IPB 300, IPB 240 and IPB 200 being used for first to third three stories, respectively. The shortest natural period of this system is 0.4076 (*sec*). It is assumed that damping value for each storey is proportional to stiffness characteristics and the proportion ratio is considered 0.001. In this example, $\Delta t = 0.1$ (*sec*) is selected as the time increment that is less than the critical range. This example is also analyzed by other methods such as Newmark method, Wilson-θ method and Quadratic B-Spline Method. The displacement time-history of the 9th storey is plotted from time *t= 0* to *t=4* second in Fig. 5. This Figure shows that the results of the proposed method being close to the results from other methods.

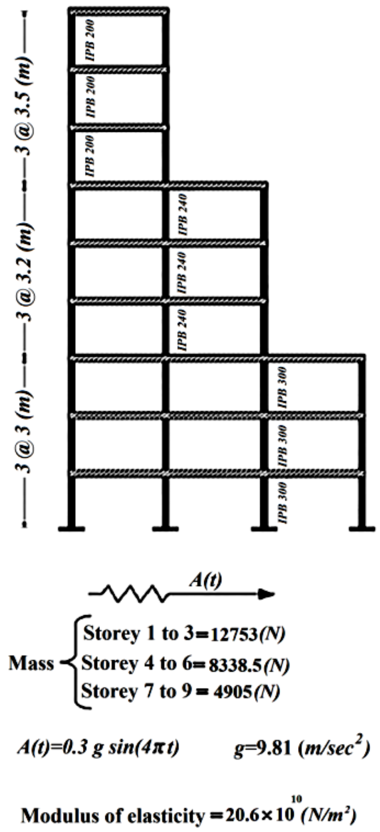

Figure 4. Model of shear building subjected to support excitation

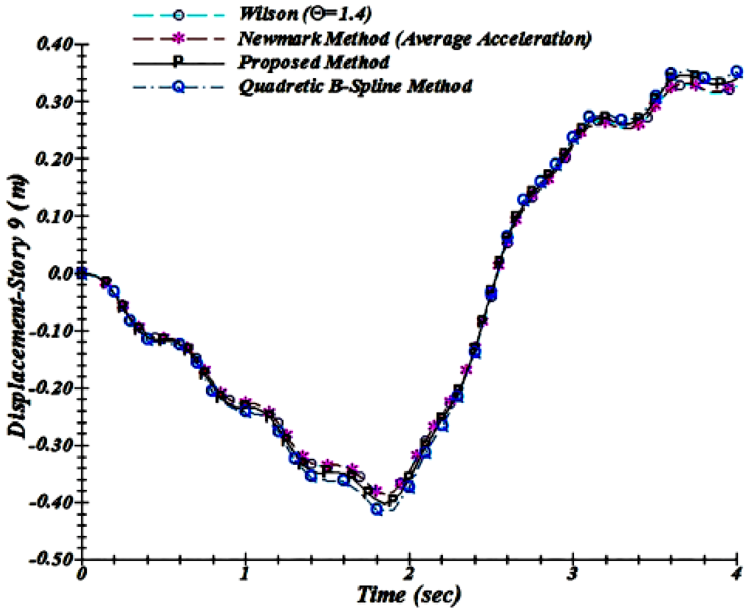

Figure 5. Displacement time-history of 9th storey shown in Fig. 4

In this example, $\Delta t = 0.1(sec)$ is selected as the time increment. The consumed time for the analysis is shown in the Table 3. It is noted that for different methods, the analysis time is computed without considering the required time for implementing input data. In fact, this time shows the required time for analyzing the system under mentioned method. As seen in Table 3, the analysis time of the proposed method is less than the others. It is another advantage of the proposed method.

Table 3: Differences between the analysis time of the methods in evaluation of displacement

Method	**Δt (sec)**	**Time Consuming to Analysis (*sec*)**
Proposed Method	$\Delta t = 0.1$	0.030973
Newmark Method (Average Acceleration)	$\Delta t = 0.1$	0.033354
Wilson-θ	$\Delta t = 0.1$	0.037063
Quadratic B-Spline Method	$\Delta t = 0.1$	0.031084

6. CONCLUSION

This paper proposes a conditionally stable explicit method for linear dynamic response analysis of SDOF and MDOF systems; which can be used as a simple method to determine the response of dynamical systems to arbitrary dynamic loads. Numerical examples show that the obtained response histories (displacement, velocity and acceleration) that are computed by the proposed method has appropriate accuracy as compared to the results from theoretical and other numerical time integration methods. Undoubtedly, using a second order polynomial to describe the acceleration, implies that a fourth order polynomial function approximates the displacement. Furthermore, since the proposed method computes six responses of the structure (i.e. two displacements, two velocities and two accelerations), the computational cost is reduced as compared to other time integration methods. Accuracy of the proposed method is appropriate in comparison to other methods. In example one, where the results are compared to those of the theoretical method, the average percentage of error in displacements and accelerations are 1.46 and 1.027, respectively. These values indicate that the accuracy of the proposed method is suitable.

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